The integrable discretization of the Bianchi–Ernst system

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Abstract

We present the constraint for the discrete Moutard equation which gives the integrable discretization of the Bianchi–Ernst system. We also derive the discrete analogue of the Bianchi transformation between solutions of such a system (the Darboux-Bäcklund transformation in soliton terminology). We finally obtain the superposition of discrete Bianchi transformations.

Key words: Discrete integrable systems, the Bianchi–Ernst system
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1 Introduction

In his studies on isometric deformation of surfaces, Bianchi considered [1] the Moutard equation [2]

$$\boldsymbol{N}_{,uv} = f\boldsymbol{N}, \tag{1}$$

supplemented by the constraint

$$\boldsymbol{N}\cdot\boldsymbol{N} = U(u) + V(v), \tag{2}$$

where $\boldsymbol{N} := (N_0, N_1, N_2)$, "$\cdot$" denotes the scalar product (Bianchi considered the case $\epsilon = 1$ only)

$$\boldsymbol{A} \cdot \boldsymbol{B} := A_0 B_0 + \epsilon (A_1 B_1 + A_2 B_2), \qquad \epsilon = \pm 1 \tag{3}$$

and $U(u)$, $V(v)$ are given funtions of single variables. In the considered reduction the function f can be given in terms of $\boldsymbol{N}$ and $U(u)$, $V(v)$ as follows

$$f = -\frac{\boldsymbol{N}_{,u} \cdot \boldsymbol{N}_{,v}}{U(u) + V(v)}. \tag{4}$$

Performing the following complex changes of the independent variables

$$u = \rho + iz, \qquad v = \rho - iz, \tag{5}$$

and of the dependent ones

$$\xi = \frac{N_1 + iN_2}{\sqrt{r} + N_0}, \qquad r = \boldsymbol{N} \cdot \boldsymbol{N}, \tag{6}$$

equations (1)–(2) are transformed into the following system

$$\begin{gathered} (\xi\bar{\xi} + \epsilon)(\xi_{,\rho\rho} + \xi_{,zz} + \tfrac{r_{,\rho}}{r}\xi_{,\rho} + \tfrac{r_{,z}}{r}\xi_{,z}) = 2\bar{\xi}\left((\xi_{,\rho})^2 + (\xi_{,z})^2\right) \\ r_{,\rho\rho} + r_{,zz} = 0, \qquad \epsilon = \pm 1. \end{gathered} \tag{7}$$

In the the case $\epsilon = -1$ the system (7) was considered by Ernst and describes axisymmetric stationary vacuum Einstein fields [3] as well as the interaction of gravitational waves [4]. Therefore we shall call the system (1)-(2) (or (7)) the Bianchi–Ernst system.

The existence of the (Darboux–Bäcklund) transformation between solutions of the Bianchi–Ernst system (1)-(2) (and therefore its integrability) was established in [1]; for the interpretation of the Bianchi–Ernst system in the context of the modern soliton theory, see [5–12]. In the simplest case $r = const$ the Bianchi–Ernst system reduces to the S^2 chiral system equivalent to the sine-Gordon equation. Therefore the Bianchi–Ernst system can also be interpreted as a non-isospectral integrable extension of a chiral model (a harmonic map or a nonlinear σ model) see e.g. [13,9,14] and references therein.

During the last few years the integrable discrete (difference) analogues of geometrically significant integrable differential equations have attracted considerable attention [15–18]. Indeed they represent the "building blocks" of a new discipline, the integrable Discrete Geometry, and are also potentially significant in a physical context. For example, the discrete analogue of the sine-Gordon equation ($r = const$), known as the Hirota equation [19], not only describes the discrete analogue of pseudospherical surfaces [20,21] and

of Chebyshev nets on a sphere [22], but also turns out to be relevant in the analysis of some solvable models of statistical mechanics and quantum field theory (see also [23,24] for other examples).

In this Letter we introduce the integrable discrete analogue of the Bianchi–Ernst system (1)-(2), namely the system

$$\begin{gathered} \boldsymbol{N}_{(12)} + \boldsymbol{N} = F(\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}), \\ (\boldsymbol{N}_{(12)} + \boldsymbol{N}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) = U(m_1) + V(m_2), \end{gathered} \tag{8}$$

where subscripts in brackets denote shifts in the discrete variables m_1 and m_2 (for details see the next section). The above system can be rewritten as the single nonlinear equation

$$\boldsymbol{N}_{(12)} + \boldsymbol{N} = \frac{U(m_1) + V(m_2)}{(\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)})} (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}). \tag{9}$$

We also establish, in the spirit of Bianchi, the integrability of the system (8) obtaining: i) the Darboux-type transformation between solutions of the system and ii) the nonlinear superposition principle for the solutions.

Remark 1 *We consider in this paper the standard (in the continuous case) situation* $U(m_1) + V(m_2) > 0$ *only.*

The interesting geometric aspects of the results of this Letter are discussed in [30,27], in the framework of the theory of discrete asymptotic nets [28,29]. The relevance of the system (8) in the discrete theory of gravity (see [31] and references therein) is an open and, in our opinion, very important theoretical problem.

The Letter is organized as follows. We first recall the necessary material concerning the discrete Moutard equation and its Darboux-type (discrete analogue of the Moutard) transformation [25,26,28,29]. Then we introduce the proper analogue of the constraint (2), which leads to the integrable reduction (8) of the discrete Moutard equation. In the next step we study consequences of such a constraint; in particular, we construct the corresponding reduction of the discrete Moutard transformation. Finally we show the permutability of the superposition of such transformations.

This work, fully inspired and motivated by the results presented in [1], is dedicated to the memory of L. Bianchi, an outstanding precursor of the modern theory of integrable systems.

2 The discrete Bianchi–Ernst system

Consider the mapping $\boldsymbol{N} : \mathbb{Z}^2 \to \mathbb{R}^3$ satisfying the discrete analogue of the Moutard equation [25]

$$\boldsymbol{N}_{(12)} + \boldsymbol{N} = F(\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}), \tag{10}$$

where $F : \mathbb{Z}^2 \to \mathbb{R}$ is a given scalar function of the discrete variables $(m_1, m_2) \in \mathbb{Z}^2$ and $\boldsymbol{N}_{(1)}(m_1, m_2) = \boldsymbol{N}(m_1 + 1, m_2)$, $\boldsymbol{N}_{(2)}(m_1, m_2) = \boldsymbol{N}(m_1, m_2 + 1)$, $\boldsymbol{N}_{(12)}(m_1, m_2) = \boldsymbol{N}(m_1+1, m_2+1)$. Given [26] a scalar solution Θ of equation (10)

$$\Theta_{(12)} + \Theta = F(\Theta_{(1)} + \Theta_{(2)}), \tag{11}$$

then the solution $\boldsymbol{N}'$ of the system of equations

$$(\boldsymbol{N}'_{(1)} \mp \boldsymbol{N}) = \frac{\Theta}{\Theta_{(1)}}(\boldsymbol{N}' \mp \boldsymbol{N}_{(1)}) \tag{12}$$

$$(\boldsymbol{N}'_{(2)} \pm \boldsymbol{N}) = \frac{\Theta}{\Theta_{(2)}}(\boldsymbol{N}' \pm \boldsymbol{N}_{(2)}) \tag{13}$$

satisfies the Moutard equation (10) with transformed potential

$$F' = \frac{\Theta_{(1)}\Theta_{(2)}}{\Theta\ \Theta_{(12)}}F. \tag{14}$$

Remark 2 *We consider [28,29] two possibilities of signs in the Moutard transformation in order: (i) to preserve the symmetry between the variables m_1 and m_2, and (ii) to reproduce the discrete Bianchi-Ernst equation in the superposition formula (see Theorem 9 for details).*

The following Lemma can be checked using equations (10)-(13).

Lemma 3 *Denote by*

$$\begin{aligned} Y &:= (\boldsymbol{N}_{(12)} + \boldsymbol{N}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}), \\ Y' &:= (\boldsymbol{N}'_{(12)} + \boldsymbol{N}') \cdot (\boldsymbol{N}'_{(1)} + \boldsymbol{N}'_{(2)}), \\ a &:= (\boldsymbol{N}'_{(1)} \mp \boldsymbol{N}) \cdot (\boldsymbol{N}' \mp \boldsymbol{N}_{(1)}), \\ b &:= (\boldsymbol{N}'_{(2)} \pm \boldsymbol{N}) \cdot (\boldsymbol{N}' \pm \boldsymbol{N}_{(2)}), \end{aligned} \tag{15}$$

where $\boldsymbol{N}$ and $\boldsymbol{N}'$ are connected by the discrete Moutard transformation (12)-(13)*, then the following identities holds*

$$Y - Y' = F\frac{\Theta_{(1)} + \Theta_{(2)}}{\Theta_{(12)}}(Y - a - b), \tag{16}$$

$$Y - a - b_{(1)} = F\frac{\Theta_{(1)}}{\Theta_{(12)}}(Y - a - b), \tag{17}$$

$$Y - a_{(2)} - b = F\frac{\Theta_{(2)}}{\Theta_{(12)}}(Y - a - b). \tag{18}$$

The following important result is a straightforward consequence of the above Lemma.

Theorem 4 *If $\boldsymbol{N}$ and $\boldsymbol{N}'$ are connected by the discrete Moutard transformation* (12)-(13)*, then the condition*

$$(\boldsymbol{N}_{(12)} + \boldsymbol{N}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) = (\boldsymbol{N}'_{(12)} + \boldsymbol{N}') \cdot (\boldsymbol{N}'_{(1)} + \boldsymbol{N}'_{(2)}) \tag{19}$$

is equivalent to the following system of three equations

$$(\boldsymbol{N}_{(12)} + \boldsymbol{N}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) = U(m_1) + V(m_2), \tag{20}$$

$$(\boldsymbol{N}'_{(1)} \mp \boldsymbol{N}) \cdot (\boldsymbol{N}' \mp \boldsymbol{N}_{(1)}) = U(m_1) \mp k, \tag{21}$$

$$(\boldsymbol{N}'_{(2)} \pm \boldsymbol{N}) \cdot (\boldsymbol{N}' \pm \boldsymbol{N}_{(2)}) = V(m_2) \pm k, \tag{22}$$

where $U(m_1)$ and $V(m_2)$ are functions of single variables only and k is a constant.

Remark 5 *Notice that the Moutard equation* (10) *subjected to the condition* (20) *is exactly the discrete Bianchi–Ernst system* (8)*, moreover the potential F is of the form*

$$F = \frac{U(m_1) + V(m_2)}{(\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}) \cdot (\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)})}. \tag{23}$$

In this paper we are going to show the integrability of the reduction (20) of the Moutard equation (10). Let us first derive some consequences of the condition (19).

Denote by $\boldsymbol{n}_0 = \boldsymbol{N}_{(12)} + \boldsymbol{N}$ (we assume that $\boldsymbol{n}_0 \cdot \boldsymbol{n}_0 > 0$) and choose a pair of unit (of length ϵ) vectors, say $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, such that $\{\boldsymbol{n}_A\}_{A=0}^2$ form an orthogonal basis. Define functions x^A, $A = 0, 1, 2$, as the coefficients of the decomposition of the vector $\Theta(2\boldsymbol{N}' \mp \boldsymbol{N}_{(1)} \pm \boldsymbol{N}_{(2)})$ in that basis, i.e,

$$\boldsymbol{N}' = \frac{1}{2}\left(\pm\boldsymbol{N}_{(1)} \mp \boldsymbol{N}_{(2)}\right) + \frac{x^A}{2\Theta}\boldsymbol{n}_A, \tag{24}$$

where the summation convention holds. In virtue of the Moutard transformation (12)-(13), the above equation can be rewritten as

$$\Theta_{(1)}(\boldsymbol{N}'_{(1)} \mp \boldsymbol{N}) + \Theta_{(2)}(\boldsymbol{N}'_{(2)} \pm \boldsymbol{N}) = x^A \boldsymbol{n}_A. \tag{25}$$

The scalar multiplication of both sides of equation (25) by $\boldsymbol{N}_{(1)} + \boldsymbol{N}_{(2)}$ gives, due to conditions (20)-(22),

$$x^0 = \frac{\mp\Theta_{(1)}(U \mp k) \pm \Theta_{(2)}(V \pm k)}{U+V}. \tag{26}$$

Let us introduce the rotation coefficients p_A^B and q_A^B by the unique decompositions

$$\boldsymbol{n}_A = p_A^B \boldsymbol{n}_{B(1)}, \qquad \boldsymbol{n}_A = q_A^B \boldsymbol{n}_{B(2)}. \tag{27}$$

The compatibility condition for the above system reads

$$q_C^A p_{A(2)}^B = p_C^A q_{A(1)}^B =: H_C^B. \tag{28}$$

Inserting the expression (24) into the discrete Moutard transformation (12)-(13) and making use of the linear independence of the vectors $\boldsymbol{n}_A$, we obtain the following linear system of equations which the functions x^A have to satisfy

$$\begin{aligned} x^A_{(1)} &= p^A_B x^B \mp \tfrac{p_0^A}{F}\Theta \pm (2p_0^A - \tfrac{\delta_0^A}{F_{(1)}})\Theta_{(1)}, \\ x^A_{(2)} &= q^A_B x^B \pm \tfrac{q_0^A}{F}\Theta \mp (2q_0^A - \tfrac{\delta_0^A}{F_{(2)}})\Theta_{(2)}, \end{aligned} \tag{29}$$

where δ_B^A is the standard Kronecker symbol.

Equations (29), (26) and (11) lead to the following linear system for the five unknowns $(\Theta, \Theta_{(1)}, \Theta_{(2)}, x^1, x^2)$

$$\begin{pmatrix} \Theta \\ \Theta_{(1)} \\ \Theta_{(2)} \\ x^1 \\ x^2 \end{pmatrix}_{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{Y_{(1)}\frac{p_0^0}{F} - b}{a_{(1)}} & \frac{bF - Y_{(1)}(\frac{Y+b}{Y}p_0^0 - \frac{1}{F_{(1)}})}{a_{(1)}} & \frac{b}{a_{(1)}}(F - \frac{Y_{(1)}}{Y}p_0^0) & \mp\frac{Y_{(1)}}{a_{(1)}}p_1^0 & \mp\frac{Y_{(1)}}{a_{(1)}}p_2^0 \\ -1 & F & F & 0 & 0 \\ \mp\frac{p_0^1}{F} & \pm\frac{Y+b}{Y}p_0^1 & \pm\frac{b}{Y}p_0^1 & p_1^1 & p_2^1 \\ \mp\frac{p_0^2}{F} & \pm\frac{Y+b}{Y}p_0^2 & \pm\frac{b}{Y}p_0^2 & p_1^2 & p_2^2 \end{pmatrix} \begin{pmatrix} \Theta \\ \Theta_{(1)} \\ \Theta_{(2)} \\ x^1 \\ x^2 \end{pmatrix}, \tag{30}$$

$$
\begin{pmatrix} \Theta \\ \Theta_{(1)} \\ \Theta_{(2)} \\ x^1 \\ x^2 \end{pmatrix}_{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & F & F & 0 & 0 \\ \frac{Y_{(2)}\frac{q_0^0}{F} - a}{b_{(2)}} & \frac{a}{b_{(2)}}\left(F - \frac{Y_{(2)}}{Y} q_0^0\right) & \frac{aF - Y_{(2)}\left(\frac{Y+a}{Y} q_0^0 - \frac{1}{F_{(2)}}\right)}{b_{(2)}} & \pm\frac{Y_{(2)}}{b_{(2)}} q_1^0 & \pm\frac{Y_{(2)}}{b_{(2)}} q_2^0 \\ \pm\frac{q_0^1}{F} & \mp\frac{a}{Y} q_0^1 & \mp\frac{a+Y}{Y} q_0^1 & q_1^1 & q_2^1 \\ \pm\frac{q_0^2}{F} & \mp\frac{a}{Y} q_0^2 & \mp\frac{a+Y}{Y} q_0^2 & q_1^2 & q_2^2 \end{pmatrix} \begin{pmatrix} \Theta \\ \Theta_{(1)} \\ \Theta_{(2)} \\ x^1 \\ x^2 \end{pmatrix}, \tag{31}
$$

where, according to the notation of Lemma 3 and the reduction under consideration,

$$
Y = U(m_1) + V(m_2), \qquad a = U(m_1) \mp k, \qquad b = V(m_2) \pm k. \tag{32}
$$

Finally, using formulas (12), (21), (24) and the orthogonality of the basis $\{\boldsymbol{n}_A\}_{A=0}^2$, one can derive the algebraic constraint satisfied by the functions Θ, $\Theta_{(1)}$, $\Theta_{(2)}$, x^1, x^2

$$
\epsilon[(x^1)^2 + (x^2)^2] + \frac{Y}{F}\Theta^2 + FY\left(-\frac{a}{Y}\Theta_{(1)} + \frac{b}{Y}\Theta_{(2)}\right)^2 - 2\Theta\left(a\Theta_{(1)} + b\Theta_{(2)}\right) = 0; \tag{33}
$$

the same constraint one obtains using, instead of equations (12), (21), equations (13) and (22).

We are ready now to present the main result of this paper. The following theorem gives the discrete analogue of the Bianchi transformation [1] between solutions of the discrete Bianchi–Ernst system (8). This transformation allows one to find, using linear steps only, new solutions of the discrete Bianchi–Ernst system (8) from given ones.

Theorem 6 *Given a solution* $\boldsymbol{N}$ *of the Bianchi–Ernst system* (8) *and the* ϵ*-unit vectors* $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ *(i.e.,* $\boldsymbol{n}_1 \cdot \boldsymbol{n}_1 = \boldsymbol{n}_1 \cdot \boldsymbol{n}_1 = \epsilon$*) orthogonal to* $\boldsymbol{N}_{(12)} + \boldsymbol{N} =: \boldsymbol{n}_0$ *and to each other, then*

(1) The linear system (30)-(31)*, where* F*,* Y*,* a*,* b *are given by equations* (23)*,* (32)*, and* p^A_B*,* q^A_B *are given by* (27)*, is compatible.*

(2) The solution $(\Theta, \Theta_{(1)}, \Theta_{(2)}, x^1, x^2)$ *of the system* (30)-(31) *satisfies the constraint* (33)*, provided that such a constraint is satisfied at the initial point.*

(3) Given the solution $(\Theta, \Theta_{(1)}, \Theta_{(2)}, x^1, x^2)$ *of the system* (30)-(31) *satisfying the constraint* (33)*, then* $\boldsymbol{N}'$*, constructed via equation* (24) *with* x^0 *given by* (26)*, is a new solution of the discrete Bianchi–Ernst system.*

PROOF. By direct, but tedious, verification. We list here only some useful identities. To prove point 1. one makes use of the following identity

$$\frac{\boldsymbol{n}_{0(12)}}{F_{(12)}} + \frac{\boldsymbol{n}_0}{F} = \boldsymbol{n}_{0(1)} + \boldsymbol{n}_{0(2)}, \tag{34}$$

valid for any solution $\boldsymbol{N}$ of the Moutard equation, from which one gets, via scalar multiplication by $\boldsymbol{n}_{0(12)}$, $\boldsymbol{n}_0$ and $\boldsymbol{n}_{0(1)} - \boldsymbol{n}_{0(2)}$,

$$\begin{aligned} \tfrac{F}{F_{(12)}} &= F(p^0_{0(2)} + q^0_{0(1)}) - H^0_0 \\ H^0_A &= \tfrac{-(U+V)\delta^0_A + p^0_A F_{(1)}(U_{(1)}+V) + q^0_A F_{(2)}(U+V_{(2)})}{(U_{(1)}+V_{(2)})} \\ q^0_{0(1)} &= p^0_{0(2)} + \tfrac{q^0_0 F_{(2)}(U+V_{(2)}) - p^0_0 F_{(1)}(U_{(1)}+V) + FF_{(1)}(U_{(1)}+V) - FF_{(2)}(U+V_{(2)})}{F(U_{(1)}+V_{(2)})} . \end{aligned} \tag{35}$$

To prove point 2. one makes use of a natural consequence of formulas (27)

$$g_{AC} = p^B_A p^D_C g_{BD(1)} = q^B_A q^D_C g_{BD(2)}, \qquad g_{AC} := \mathrm{diag}\,[(U+V)F, \epsilon, \epsilon] \,. \tag{36}$$

The proof of point 3. splits naturally into two parts. First, using point 1., we prove that such $\boldsymbol{N}'$ is related to $\boldsymbol{N}$ by the Moutard transformation (12)-(13) via the function Θ. Then, using the constraint (33), we immediately check that $\boldsymbol{N}'$ is subjected to

$$(\boldsymbol{N}'_{(12)} + \boldsymbol{N}') \cdot (\boldsymbol{N}'_{(1)} + \boldsymbol{N}'_{(2)}) = U(m_1) + V(m_2). \qquad \square \tag{37}$$

Remark 7 *The parameter* k*, present in the linear system* (30)-(31)*, is called the transformation parameter.*

Remark 8 *The linear system* (30)-(31) *can also be interpreted as a nonstandard Lax pair (zero curvature representation) of the discrete Bianchi-Ernst system* (8)*, with spectral parameter* k*. A more traditional Lax pair for system* (8) *will be presented in the forthcoming paper [27].*

Finally, we present the superposition law associated with the transformation described in Theorem 6. The proof, omitted for the sake of brevity (it will be published in [32]), consists in proving that the discrete Bianchi constraint (20) is compatible with the superposition law of the discrete Moutard transformations (12)-(13), given in [29,28].

Theorem 9 *Given a solution* $\boldsymbol{N}$ *of the Bianchi–Ernst system* (8) *and given two transforms of it (the upper index denotes the transformation!): the upper-sign transform* $\boldsymbol{N}^{(1)}$*, with the transformation parameter* k^1*, and the lower-sign transform* $\boldsymbol{N}^{(2)}$*, with the transformation parameter* k^2*. Then there exists the unique solution* $\boldsymbol{N}^{(12)}$ *of the Bianchi–Ernst system, given in algebraic terms by*

$$\boldsymbol{N}^{(12)} = -\boldsymbol{N} + \frac{k^1 + k^2}{(\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}) \cdot (\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)})}(\boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}), \tag{38}$$

which is simultaneously the lower-sign transform of $\boldsymbol{N}^{(1)}$*, with the transformation parameter* k^2*, and the upper-sign transform of* $\boldsymbol{N}^{(2)}$*, with the transformation parameter* k^1*.*

Remark 10 *Notice that the superposition formula* (38) *for the Bianchi–Ernst system* (8) *reproduces the Bianchi–Ernst system itself, after replacing the upper transformation idices by the lower translation ones.*

Remark 11 *Exactly like it was done in the continuous case [1,14] by taking* $\boldsymbol{N} \in \mathbb{R}^n$ *the above considerations can be generalized to the multicomponent discrete Bianchi–Ernst system.*

References

[1] L. Bianchi, Mem. Soc. It. delle Sc. 13 (1905) 261.

[2] Th-F Moutard, J. Ec. Pol. 45 (1878) 1.

[3] F.J. Ernst, Phys. Rev. 167 (1968) 1175.

[4] S. Chandrasekhar, V. Ferrari, Proc. Roy. Soc. London A 396 (1984) 55.

[5] D. Maison, Phys. Rev. Lett. 41 (1978) 521.

[6] V.A. Belinski, V.E. Zakharov, Zh. eksper. teor. Fiz. 75 (1978) 1953; 77 (1979) 3.

[7] G. Neugebauer, J. Phys. A 12 (1979) L67.

[8] B. Cenkl, Physica 18D (1986) 217.

[9] D. Levi, A. Sym, Phys. Lett. A 149 (1990) 381.

[10] D. Korotkin, Sfb 288 preprint 116 (1994).

[11] W.K. Schief, C. Rogers, M.E. Johnston, Chaos, Solitons and Fractals 5 (1995) 25.

[12] J. Tafel, J. Geom. Phys. 17 (1995) 381.

[13] P.O. Mazur, Acta Phys. Pol. B14 (1983) 219.

[14] M. Nieszporski, Phys. Lett. A 272 (2000) 74.

[15] A.I. Bobenko and U. Pinkall, Discretization of surfaces and integrable systems, in: Discrete integrable geometry and physics, Clarendon Press, Oxford 1999.

[16] A. Doliwa and P.M. Santini, Phys. Lett. A 233 (1997) 365-372.

[17] A. Doliwa, P.M. Santini and M. Mañas, J. Math. Phys. 41 (2000) 944-990.

[18] B.G. Konopelchenko and W.K. Schief, P. Roy. Soc. Lond. A Mat. 454 (1998) 3075.

[19] R. Hirota, J. Phys. Soc. Jpn. 43 (1977) 2079.

[20] R. Sauer, Differenzengeometrie, Springer Verlag, Berlin, 1970.

[21] A. Bobenko i U. Pinkall, J. Diff. Geom. 43 (1996) 527.

[22] A. Doliwa i P. M. Santini, [in:] *Symmetries and Integrability of Difference Equations*, D. Levi, L. Vinet and P. Winternitz (eds.), CRM Proceedings and Lecture Notes, vol. 9, p. 91, AMS, Providence.

[23] V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, Quantum Inverse Scattering Method and Correlation Functions, Cambridge University Press, Cambridge, 1993.

[24] M. Jimbo, T. Miwa, Algebraic Analysis of Solvable Lattice Models, AMS, Providence, 1995.

[25] J.J.C. Nimmo, W.K. Schief, Proc. R. Soc. London A 453 (1997) 255.

[26] W.K. Schief, in: Symmetries and Integrability of Difference Equations, Cambridge University Press, 1999, p. 137.

[27] P.M. Santini, A. Doliwa, M. Nieszporski, in preparation.

[28] A. Doliwa, solv-int/990915, Discrete asymptotic nets and W-congruences in Plücker line geometry, to be published in J. Geom. Phys.

[29] M. Nieszporski, On a discretization of asymptotic nets, submitted

[30] M. Nieszporski, Weingarten congruences as a source of integrable systems, Ph. D. thesis [in Polish].

[31] T. Regge, R. M. Williams, J. Math. Phys. 41 (2000) 3964.

[32] A. Doliwa, M. Nieszporski, P. M. Santini Discrete asymptotic nets and their integrable reductions I: the Bianchi and the Fubini–Ragazzi lattices, in preparation.